6.2 Orthogonal Sets 6.3 Orthogonal Projections

We already saw that

$$S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}0\\5\\0\end{bmatrix} \right\}$$

is such that

$$\vec{v}_1 \cdot \vec{v}_2 = 0, \quad \vec{v}_1 \cdot \vec{v}_3 = 0, \text{ and } \vec{v}_2 \cdot \vec{v}_3 = 0$$

Thus, the set S is called an orthogonal set.

Def. The set $S=\{\vec{v}_1, .., \vec{v}_p\} \subset \mathbb{R}^n$ is <u>an orthogonal set</u>

if $\boxed{v_i \cdot v_j = 0, \quad i \neq j, \quad i,j=1..,p.}$

Thm 4. $S=\{\vec{v}_1, .., \vec{v}_p\} \subset \mathbb{R}^n$, $\vec{v}_i \neq \vec{0}$, $i=1,..,p$

i) If S is an orthogonal set, then S is linearly independent. and S is a basis for Span(S).

Proof. Want to prove:

If $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0} \Rightarrow c_1 = 0, .., c_p = 0.$

Answ: If $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$

Then

$$0 = \vec{v}_1 \cdot \vec{0} = \vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p) = \vec{v}_1 \cdot (c_1\vec{v}_1) + \vec{v}_1 \cdot (c_2\vec{v}_2) + \dots + \vec{v}_1 \cdot (c_p\vec{v}_p)$$

$$\overset{\text{Thm 1}}{=} c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + \dots + c_p(\vec{v}_1 \cdot \vec{v}_p) \overset{\text{Hyp orthog.}}{=} c_1(\vec{v}_1 \cdot \vec{v}_1)$$

$$\Rightarrow \boxed{c_1(\vec{v}_1 \cdot \vec{v}_1) = 0 \Rightarrow c_1 = 0,} \text{ Since } \vec{v}_1 \neq \vec{0}.$$

A Similar procedure, using $\vec{v}_2, \vec{v}_3 \dots, \vec{v}_{p-1}$ Succesively leads to

$$0 = \vec{v}_j \cdot \vec{0} = v_j \cdot (c_1 \vec{v}_1 + \dots + c_j \vec{v}_j + \dots + c_p \vec{v}_p) =$$

$$= c_1 (v_j \cdot \vec{v}_1) + \dots + c_j (\vec{v}_j \cdot \vec{v}_j) + \dots + c_p (\vec{v}_j \cdot \vec{v}_p) \stackrel{\text{Hyp. orthog.}}{=\!=\!=}$$

$$= c_j (\vec{v}_j \cdot \vec{v}_j) \Rightarrow \boxed{c_j (\vec{v}_j \cdot \vec{v}_j) = 0 \Rightarrow c_j = 0, \text{ since } \vec{v}_j \neq 0 \quad j = 2, \dots, p-1.}$$

For $j = p$, it is completely analogous.

Since $c_1 = 0, \dots, c_p = 0$, then S is linearly independent.

Also, S spans $\text{Span}(S)$ then, S is a basis for $\text{Span}(S)$.

__Def.__ $B = \{\vec{v}_1, \dots, \vec{v}_r\} \subset \mathbb{R}^n$ is formed by orthogonal vectors (nonzero) and $\text{Span}(B) = W$ then <u>B is called an orthogonal basis for the subspace W.</u>

<u>Our Example</u>:

Since $S = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}} \right\}$ is an orthogonal set of nonzero vectors then S is linearly independent (using thm 4)

Also, S is a basis for $\mathbb{R}^3$ because S consists of 3 vectors and $\dim(\mathbb{R}^3) = 3$.

Advantage of an orthogonal basis.

Find the coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad (3.1)$$

We want to find the coefficients c_1, c_2, c_3. They will be the coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis B. It means

$$[\vec{u}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The equation (3.1) is equivalent to solving the linear system

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

row-reducing augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ -1 & 2 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

$\Rightarrow \quad c_1 = -1, \quad c_2 = 1, \quad c_3 = 2/5.$

Thus,
$$[\vec{u}]_B = \overset{\vec{u}}{\begin{bmatrix}1\\2\\3\end{bmatrix}_B} = \begin{bmatrix}-1\\1\\2/5\end{bmatrix}\begin{matrix}=c_1\\=c_2\\=c_3\end{matrix}$$

These coefficients can be obtained easily using the orthogonal property of the vectors of B.

In fact,

If $$\underbrace{c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}2\\0\\2\end{bmatrix}+c_3\begin{bmatrix}0\\5\\0\end{bmatrix}}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

then $$\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\left(\quad\swarrow\quad\right)=\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$$

thm 1

$$\Rightarrow \quad c_1\Big(\overbrace{[1\ 0\ -1]\begin{bmatrix}1\\0\\-1\end{bmatrix}}^{\vec{v}_1\cdot\vec{v}_1}\Big)=\overbrace{[1\ 0\ -1]\begin{bmatrix}1\\2\\3\end{bmatrix}}^{\vec{v}_1\cdot\vec{u}}$$

or $$2c_1 = 1-3=-2 \Rightarrow \boxed{c_1=-1}$$ Same as before.

In Symbols,

$$c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{u}$$

multiplying both sides by $\vec{v}_1$ (dot product).

$$\vec{v}_1\cdot\left(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3\right)=\vec{v}_1\cdot\vec{u}$$

Using orthogonality and thm 1

$$c_1\left(\vec{v}_1\cdot\vec{v}_1\right)=\vec{v}_1\cdot\vec{u}$$

$$\Rightarrow \quad \boxed{c_1=\frac{\vec{v}_1\cdot\vec{u}}{\vec{v}_1\cdot\vec{v}_1}}$$

Ask students to compute c_2, c_3.

Analogously,

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{[2\ 0\ 2]\begin{bmatrix}1\\2\\3\end{bmatrix}}{[2\ 0\ 2]\begin{bmatrix}2\\0\\2\end{bmatrix}} = \frac{8}{8} = 1$$

and

$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{[0\ 5\ 0]\begin{bmatrix}1\\2\\3\end{bmatrix}}{[0\ 5\ 0]\begin{bmatrix}0\\5\\0\end{bmatrix}} = \frac{10}{25} = \frac{2}{5}$$

$$\boxed{[\vec{u}]_B = \begin{bmatrix}1\\2\\3\end{bmatrix}_B = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-1\\1\\2/5\end{bmatrix}}$$

Same as before, but computation is much easier.

<u>Verification:</u>

$$(-1)\begin{bmatrix}1\\0\\-1\end{bmatrix} + (1)\begin{bmatrix}2\\0\\2\end{bmatrix} + \frac{2}{5}\begin{bmatrix}0\\5\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \checkmark$$

This result can be generalized to $\mathbb{R}^n$

<u>Thm 5.</u> $B = \{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\} \subset \mathbb{R}^n$ is an orthogonal basis for $W \subset \mathbb{R}^n$.

The coordinates of $\vec{u} \in W$ with respect to the basis B satisfy

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{u}}$$

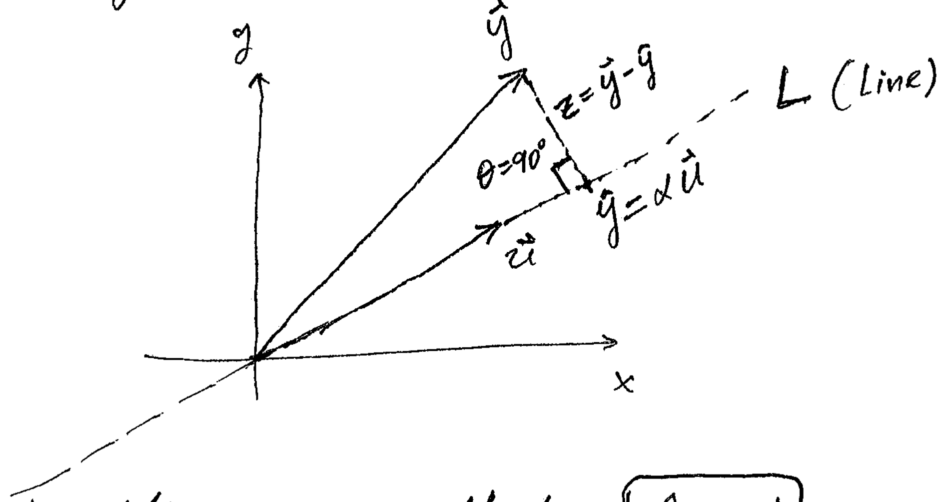
where

$$\boxed{c_j = \frac{\vec{u} \cdot v_j}{v_j \cdot v_j}, \quad j = 1, \dots, p.}$$

<u>Proof</u>:- Completely analogous to the previous computation in $\mathbb{R}^3$.

Orthogonal Projections

Consider $\vec{y}, \vec{u} \in \mathbb{R}^2$ as in the graph below



We want to obtain α such that $\boxed{\hat{y} = \alpha \vec{u}}$ and $\boxed{\vec{z} = \vec{y} - \hat{y}}$ is orthogonal to $\vec{u}$. (6.1)

From our previous results and definitions

$\vec{z} = \vec{y} - \hat{y}$ is orthogonal to $\vec{u}$ if and only if

$$(\vec{y} - \hat{y}) \cdot \vec{u} = 0$$

This implies

$$0 = (\vec{y} - \alpha \vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - (\alpha \vec{u}) \cdot \vec{u} \overset{\text{Thm 1}}{=} \vec{y} \cdot \vec{u} - \alpha (\vec{u} \cdot \vec{u})$$

$$\Rightarrow \quad \alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

or

$$\boxed{\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}} \qquad (6.2)$$

Combining (6.1) and (6.2), we have obtained a decomposition of $\vec{y} \in \mathbb{R}^2$ as

$$\boxed{\vec{y} = \hat{y} + \vec{z}} \tag{7.1}$$

where $\hat{y} = \left(\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}\right) \in \operatorname{Span}\{\vec{u}\}$ and $z \in (\operatorname{Span}\{\vec{u}\})^{\perp}$

<u>Definition</u> The vector $\hat{y}$ given by

$$\boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$$

is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$

It is also denoted as

$$\boxed{\hat{y} = \operatorname{proj}_{\vec{u}} \vec{y}}$$

As it will be shown later this decomposition (7.1) is unique.

<u>Exercise</u> For the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Find the unique decomposition

$$\vec{u} = \hat{u} + \vec{z}$$

where $\hat{u} = \text{proj}_{\vec{v}_3} \vec{u}$ and z is orthogonal to $\vec{v}_3$.

<u>Answ:</u>

According to (7.1)

$$\hat{u} = \text{proj}_{\vec{v}_3} \vec{u} = \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3 = \frac{[0\ 5\ 0] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{[0\ 5\ 0] \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$= \frac{2}{5} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

and

$$\vec{z} = \vec{u} - \hat{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Therefore, the decomposition of $\vec{u}$ is as follows:

$$\vec{u} = \underset{\hat{u}}{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}} + \underset{\vec{z}}{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Verification that $\vec{z}$ is orthogonal to $\vec{v}_3$

$$\vec{z} \cdot \vec{v}_3 = [1\ 0\ 3] \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = 0 \checkmark$$

~~And~~ $\vec{z} = \vec{u} - \hat{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Verification of $\vec{z}$ orthogonal to $\vec{v}_3$

$$\vec{z} \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = [1\ 0\ 3] \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = 0 \checkmark$$

or

$$\boxed{\begin{array}{c} \vec{u} = \hat{u} + \vec{z} \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{array}}$$

The concept of orthogonal projection can be extended to any subspace $W \subset \mathbb{R}^n$ (It does not have to be a line along $\vec{u}$).

<u>Thm 8</u> (Orthogonal Decomposition Theorem)

$W \subset \mathbb{R}^n$ subspace and $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_p\}$ orthogonal basis of W.

Then,

Any $\vec{y} \in \mathbb{R}^n$ can be written uniquely as

$$\boxed{\vec{y} = \hat{y} + \vec{z}} \qquad (8.1)$$

where $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$

and
$$\boxed{\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p = \operatorname{proj}_W \vec{y}} \qquad (8.2)$$

The concept of orthogonal projection can be extended to any subspace $W \subset \mathbb{R}^n$ (It does not have to be a line along $\vec{u}$).

<u>Thm 8</u> (Orthogonal Decomposition Theorem)

$W \subset \mathbb{R}^n$ subspace and $B = \{\vec{u}_1, .., \vec{u}_p\}$ orthogonal basis of W.

Then,

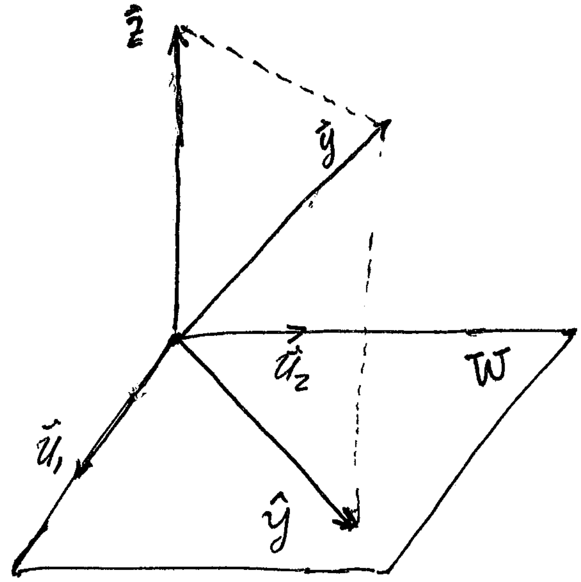
Any $\vec{y} \in \mathbb{R}^n$ can be written uniquely as

$$\boxed{\vec{y} = \hat{y} + \vec{z}} \qquad (8.1)$$

where $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$

and

$$\boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p = \text{proj}_W \vec{y}} \qquad (8.2)$$



$$\vec{y} = \hat{y} + \vec{z}$$

<u>Proof.</u>– Consider an arbitrary $\vec{y} \in \mathbb{R}^n$

and define
$$\hat{y} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_p \vec{u}_p \in W \quad (9.1)$$

and
$$\vec{z} = \vec{y} - \hat{y}. \quad (9.2)$$

Since we want $\vec{z} \in W^{\perp}$, we require $\vec{z}$ to be orthogonal to all $\vec{u}_j$ $(j=1,..,p)$ (This is enough according to previous thm)

Thus,
$$(\vec{z} \cdot \vec{u}_j) = ((\vec{y} - \hat{y}) \cdot \vec{u}_j) = 0, \quad j=1,..,p.$$

or
$$\vec{y} \cdot \vec{u}_j - ((\alpha_1 \vec{u}_1 + \cdots + \alpha_j \vec{u}_j + \cdots + \alpha_p \vec{u}_p) \cdot \vec{u}_j) = 0$$

using orthogonality
$$\alpha_j (\vec{u}_j \cdot \vec{u}_j) = \hat{y} \cdot \vec{u}_j$$

$$\Rightarrow \boxed{\alpha_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j},} \quad j = 1, 2, .., p. \quad (9.3)$$

Therefore,

$\vec{z} = \vec{y} - \hat{y}$ is orthogonal to W

if $$\boxed{\hat{y} = \frac{\vec{y}\cdot u_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p = \text{proj}_W\vec{y}} \quad (10.1)$$

As a Consequence,

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W$ is given by (10.1) and $\vec{z} = \vec{y} - \hat{y} \in W^\perp$.

<u>Uniqueness</u> If there exists $\hat{y}_1 \in W$ and $\vec{z}_1 \in W^\perp$

Such that $\vec{y} = \hat{y}_1 + \vec{z}_1$

then $\hat{y}_1 + \vec{z}_1 = \vec{y} = \hat{y} + \vec{z}$

$$\Rightarrow \hat{y}_1 - \hat{y} = \vec{z} - \vec{z}_1 \Rightarrow \hat{y}_1 - \hat{y} \in W \cap W^\perp$$

$$\Rightarrow \hat{y}_1 - \hat{y} = \vec{0} \Rightarrow \underline{\hat{y}_1 = \hat{y}}$$

$$\Rightarrow \vec{z} - \vec{z}_1 = 0 \Rightarrow \underline{\vec{z}_1 = \vec{z}}$$

6.3 Exerc. #3

$$\vec{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

Verify $\{\vec{u}_1, \vec{u}_2\} \subset \mathbb{R}^3$ is an orthogonal set and find the orthogonal projection of $\vec{y}$ onto $W = \operatorname{Span}\{\vec{u}_1, \vec{u}_2\}$.

Answ:

$$\vec{u}_1 \cdot \vec{u}_2 = [1\ 1\ 0]\begin{bmatrix}-1\\1\\0\end{bmatrix} = -1+1 = 0. \checkmark$$

According to thm 8

$$\operatorname{proj}_W \vec{y} = \hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 =$$

$$= \frac{[-1\ 4\ 3]\begin{bmatrix}1\\1\\0\end{bmatrix}}{[1\ 1\ 0]\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{[-1\ 4\ 3]\begin{bmatrix}-1\\1\\0\end{bmatrix}}{[-1\ 1\ 0]\begin{bmatrix}-1\\1\\0\end{bmatrix}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$\Longrightarrow$

$$\operatorname{proj}_W \vec{y} = \hat{y} = \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{5}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\4\\0\end{bmatrix}$$

and

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}-1\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

and

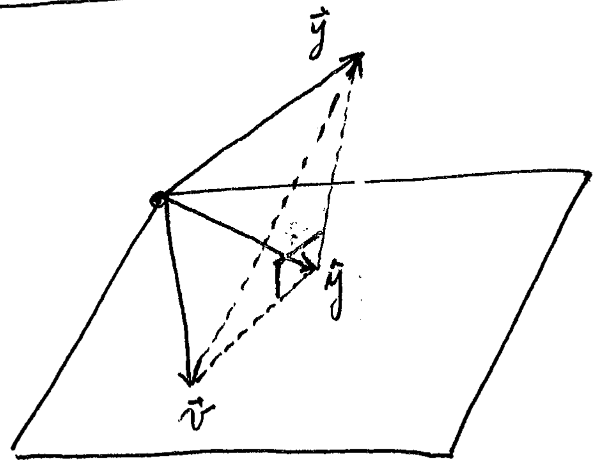
$$\vec{y} = \overset{\in W}{\begin{bmatrix}-1\\2\\0\end{bmatrix}} + \overset{\in W^{\perp}}{\begin{bmatrix}0\\0\\3\end{bmatrix}}$$

An obvious property of the orthogonal projection is that

If $\vec{y} \in W$ then $\hat{y} = \text{proj}_W \vec{y} = \vec{y}$.

(why?.)

Best Approximation Theorem



We want to compare

$$dist(\vec{y}, \hat{y}) = \|\vec{y} - \hat{y}\|$$

with the

$$dist(\vec{y}, \vec{v}) = \|\vec{y} - \vec{v}\|, \quad \text{for } \vec{v} \in W, \ \vec{v} \neq \hat{y}.$$

We are looking to the best approximation to $\vec{y}$ by vectors in W.

Theorem 9.

1) $W \subset \mathbb{R}^n$
2) $\vec{y}$ an arbitrary vector in $\mathbb{R}^n$.
3) $\hat{y} = \text{proj}_W \vec{y}$

Then, $\hat{y}$ is the closest point in W to $\vec{y}$ or

$$(13.1) \qquad \boxed{\|\vec{y}-\hat{y}\| < \|\vec{y}-\vec{v}\|,} \quad \text{for all } \vec{v} \in W,\ \vec{v} \neq \hat{y}.$$

Proof.

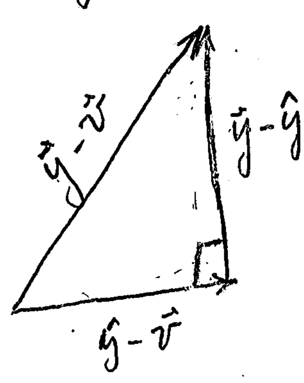
We will show the inequality for the square of the two sides is true and (13.1) will follow from this.

Since $\vec{v} \in W$ and $\hat{y} \in W \Rightarrow \hat{y} - \vec{v} \in W$

Also, $\vec{z} = \vec{y} - \hat{y} \in W^{\perp}$ (thm 8).

Therefore, $\vec{y} - \hat{y}$ is orthogonal to $\hat{y} - \vec{v}$.

Now, $\vec{y} - \vec{v} = (\vec{y} - \hat{y}) + (\hat{y} - \vec{v})$



$$\overset{\text{Pythagorean Thm}}{\Longrightarrow} \quad \|\vec{y}-\vec{v}\|^2 = \|\vec{y}-\hat{y}\|^2 + \|\hat{y}-\vec{v}\|^2$$

Since $\vec{v} \neq \hat{y}$ $\quad \|\hat{y}-\vec{v}\| > 0$

$$\Rightarrow \quad \|\vec{y}-\vec{v}\|^2 > \|\vec{y}-\hat{y}\|^2 \overset{\sqrt{\ }}{\Longrightarrow} \|y-\hat{y}\| < \|y-\vec{v}\|,$$

The distance from $\vec{y}$ to $\vec{v}$

$$\|\vec{y}-\vec{v}\|$$

is considered as the error of using $\vec{v}$ in W as approximation of $\vec{y}$.

Theorem 9 states that the minimum error is when $\vec{v}=\hat{y}=\mathrm{proj}_W \vec{y}$.

Section 6.3 #15)

Find the distance from $\vec{y}=\begin{bmatrix}5\\-9\\5\end{bmatrix}$ to the plane in $\mathbb{R}^3$ spanned by $\vec{u}_1=\begin{bmatrix}-3\\-5\\1\end{bmatrix}$ and $\vec{u}_2=\begin{bmatrix}-3\\2\\1\end{bmatrix}$

Answ: we will find $\hat{y}$ and then compute the distant $\|\vec{y}-\hat{y}\|$.

First, a check that $\vec{u}_1$ is orthogonal to $\vec{u}_2$.

$$\vec{u}_1\cdot\vec{u}_2=[-3\ -5\ 1]\begin{bmatrix}-3\\2\\1\end{bmatrix}=9-10+1=0\ \checkmark$$

$\hat{y}=\mathrm{proj}_W \vec{y}$, where $W=\mathrm{span}\{u_1,u_2\}$.

$$\hat{y}=c_1\vec{u}_1+c_2\vec{u}_2$$

$$c_1=\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}=\frac{[5\ -9\ 5]\begin{bmatrix}-3\\-5\\1\end{bmatrix}}{[-3\ -5\ 1]\begin{bmatrix}-3\\-5\\1\end{bmatrix}}=$$

$$\Rightarrow c_1=\frac{35}{35}=1$$

$$c_2=\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}=\frac{[5\ -9\ 5]\begin{bmatrix}-3\\2\\1\end{bmatrix}}{[-3\ 2\ 1]\begin{bmatrix}-3\\2\\1\end{bmatrix}}=\frac{-28}{14}=-2$$

$$\Rightarrow \hat{y}=\vec{u}_1-2\vec{u}_2=\begin{bmatrix}-3\\-5\\1\end{bmatrix}-2\begin{bmatrix}-3\\2\\1\end{bmatrix}=$$

$$\Rightarrow \boxed{\hat{y}=\begin{bmatrix}3\\-9\\-1\end{bmatrix}} \Rightarrow \boxed{\vec{y}-\hat{y}=\begin{bmatrix}2\\0\\6\end{bmatrix}=\vec{z}}$$

$$\Rightarrow \mathrm{dist}=\|\vec{y}-\hat{y}\|=\sqrt{4+36}=\boxed{\sqrt{40}}.$$